

Formal Solutions of a Class of Pfaffian Systems in Two Variables*

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Abstract

In this article, we present an algorithm which computes a fundamental matrix of formal solutions of completely integrable Pfaffian systems with normal crossings in two variables, based on [Barkatou, 1997]. A first step was set in [Barkatou et al., 2006] where the problem of rank reduction was tackled via the approach in [Levelt, 1991]. We give instead a Moser-based approach following [Barkatou et al., 2014; Barkatou, 1995]. And, as a complementary step, we associate to our problem a system of ordinary linear singular differential equations from which the formal invariants can be efficiently derived via the package ISOLDE [Barkatou et al., 2013], implemented in the computer algebra system Maple.

Keywords: Linear systems of partial differential equations, Pfaffian systems, Formal solutions, Moser-based reduction, Hukuhara-Turritin's normal form

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1 Introduction

Let $\mathcal{O} = \mathbb{C}[[x_1, x_2, \dots, x_m]]$ be the ring of formal power series in $x = (x_1, x_2, \dots, x_m)$ over the field of complex numbers \mathbb{C} and K its field of fractions. Let V be a K -vector space of dimension n , $A^{(1)}, \dots, A^{(m)}$ be nonzero matrices with entries in \mathcal{O} , and p_1, \dots, p_m be nonnegative integers. Set $\delta_i = x_i \frac{\partial}{\partial x_i}$. Then the operator $\Delta_i = \delta_i - A^{(i)}/x_i^{p_i}$, $i = 1, \dots, m$, is a δ_i -differential operator acting on V , that is, an additive map from V to itself satisfying the Leibniz condition:

$$\forall f \in \mathcal{O}, v \in V, \Delta_i(fv) = \delta_i(f)v + f\Delta_i(v).$$

Let Y be an unknown n -dimensional column vector. In this article, we are interested in the formal reduction of the so-called *completely integrable Pfaffian systems with normal crossings*, the class of linear systems of partial differential equations in m variables and dimension n , given by

$$\Delta_i Y = 0, \quad 1 \leq i \leq m. \quad (1)$$

satisfying the integrability conditions (pairwise commutativity of the operators),

$$\Delta_i \circ \Delta_j = \Delta_j \circ \Delta_i, \quad 1 \leq i, j \leq m.$$

Pfaffian systems arise in many application [Awane, 2000] including the studies of aerospace and celestial mechanics [Broucke, 1978]. By far, the most important for applications are those with normal crossings [Novikov et al., 2002]. The associated pfaff (1-form) to system (1) is $d = \sum_{i=1}^m \frac{A^{(i)}}{x_i^{p_i+1}} dx_i$. Without loss of generality, the singularity of system(1) is placed at the origin. Otherwise, translations in the independent variables can be performed. The singular locus of the system is the union of hyperplanes of coordinates $x_1 x_2 \dots x_m = 0$. This is what is referred to as *normal crossings*.

Let $T \in GL(V)$. A change of basis $Y = TZ$ gives rise to

$$\tilde{\Delta}_i Z = 0, \quad 1 \leq i \leq m \quad (2)$$

where $\tilde{\Delta}_i = T^{-1} \Delta_i T = \delta_i - \tilde{A}^{(i)}/x_i^{\tilde{p}_i}$ and

$$\frac{\tilde{A}^{(i)}}{x_i^{\tilde{p}_i}} = T^{-1} \left(\frac{A^{(i)}}{x_i^{p_i}} T - \delta_i T \right), \quad 1 \leq i \leq m. \quad (3)$$

We say that system (2) is *equivalent* to system (1). Without loss of generality, we assume that $A^{(i)}(x_i = 0)$ are nonzero matrices; otherwise p_i can be readjusted. The m -tuple (p_1, \dots, p_m) of nonnegative integers is called the *Poincaré rank* of the system. It is said to be minimal, and is called the *true Poincaré rank*, whenever the p_i 's are simultaneously and individually the smallest among all the possible changes of basis $Y = TZ$, $T \in GL_n(K)$.

In [Charrière et al., 1981; van den Essen et al., 1982], the language of stable modules over the ring of power series is used to establish the following theorem (Theorem 1 in [Charrière et al., 1981] and Main Theorem in [van den Essen et al., 1982]).

Theorem 1 Consider the completely integrable system (1). Under additional ramifications, $x_i = t_i^{\alpha_i}$ where α_i is a positive integer, $1 \leq i \leq m$, there exists T that belongs to $GL_n(\mathbb{C}((t_1, \dots, t_m)))$ such that the change of basis $Y = TZ$ gives the equivalent system:

$$T^{-1}\Delta_i T = \frac{\partial}{\partial t_i} - \tilde{A}^{(i)}(t_i)$$

where $\tilde{A}^{(i)}(t_i) = \text{Diag}(\tilde{A}_1^i, \tilde{A}_2^i, \dots, \tilde{A}_h^i)$, and for all $1 \leq k \leq h$ we have

- $\tilde{A}_k^i = w_k^i(t_i^{-1})I_{n_k} + t_i^{-1}N_k^i$;
- $w_k^i(t_i) = \sum_{j=1}^{\alpha_i p_i + 1} \xi_{ijk} t_i^{-j}$ is a polynomial in t_i^{-1} with coefficients in \mathbb{C} . In particular, $\bar{w}_k^i(t_i) = \sum_{j=2}^{\alpha_i p_i + 1} \xi_{ijk} t_i^{-j}$, for all $1 \leq k \leq h$, are called the x_i -exponential parts;
- N_k^i is a square matrix of order n_k with elements in \mathbb{C} , nilpotent in upper triangular form;
- $n_1 + n_2 + \dots + n_h = n$.

This theorem guarantees the existence of a change of basis which takes system (1) to Hukuhara-Turritin's normal form from which the construction of a fundamental matrix of formal solutions (see (6) for $m = 2$) is straightforward. In particular, if two systems are *equivalent* then they have the same x_i -exponential parts for all $1 \leq i \leq m$. However, the formal reduction, that is to say the algorithmic procedure computing such a change of basis, is a question of another nature.

We remark that the integrability conditions and the property of normal crossings play a major role in establishing this theorem. In particular, they give rise to the following aspect of system (1) (Proposition 1, page 8, [van den Essen et al., 1982]).

Proposition 1 The eigenvalues of $A^{(i)}(x_i = 0)$, $1 \leq i \leq m$, lie in the field of constants.

The univariate case, $m = 1$, is the well-known case of linear singular system of ordinary differential equations (ODS, in short) which have been studied extensively (see, e.g., [Balser, 2000; Wasow, 2002] and references therein). Moreover, unlike the case of $m > 1$, algorithms to related problems leading to the construction of the formal solutions and computation of the *exponential parts* have been developed by various authors (see, e.g., [Barkatou, 1997; Barkatou et al., 2013, 1999, 2009] and references therein). The package ISOLDE [Barkatou et al., 2013] written in the computer algebra system Maple is dedicated to the symbolic resolution of ODS and more generally linear functional matrix equations.

The bivariate case, $m = 2$, is the interest of this article. Our first contribution is an explicit method to compute the x_i -*exponential parts*. Alongside their importance in the asymptotic theory, their computation reduces the ultimate task of formal reduction to constructing a basis of the \mathbb{C} -space of regular solutions (see, for $m = 1$, [Barkatou et al., 1999]). Rank reduction as well can complement our work into full formal reduction. Rank reduction is the explicit computation of a change of basis which takes system (1) to an *equivalent* system whose rank is the true Poincaré rank. Two well-known approaches in the univariate case are those of Levelt [Levelt, 1991] and Moser [Moser, 1960]. The former is generalized for systems (1) in two variables ($m = 2$) in [Barkatou et al., 2006]. The latter, on the other hand, gives a *reduction criterion* upon which the efficient algorithms of [Barkatou et al., 2013] are based. It is generalized for singularly-perturbed linear differential systems in [Barkatou et al., 2014] and examples over the bivariate field favored its efficiency [Section 4 in [Barkatou et al., 2014]]. Our second contribution is a Moser-based rank reduction algorithm of system (1) in two variables. This establishes its formal reduction upon following the univariate-case description of [Barkatou, 1997].

We remark that the multivariate particular case of $p_i = 0$ for all $1 \leq i \leq m$ (systems with singularity of first kind) is studied in [LeRoux, 2006] and references therein.

Our article is organized as follows: In Section 2 we give the preliminaries and restrict our notations to two variables. In Section 3 we review the formal reduction techniques in [Barkatou, 1997] of ODS, including the computation of the *exponential parts*. Thereafter the discussion is restricted to the bivariate case. In Section 4 we present the building blocks of formal reduction reducing the discussion to two elements. The first of which is the computation of the x_i -*exponential parts* given in Section 5. The second is the Moser-based rank reduction algorithm given in Section 6. We finally discuss the limitations and point out prospects of further investigations in Section 7. Our main results are Theorem 3 and Theorem 4.

2 Preliminaries and Notations

For the clarity of presentation in the bivariate case, we set $\mathcal{O} = \mathbb{C}[[x, y]]$, K its field of fractions, $\mathcal{O}_x = \mathbb{C}[[x]]$, and $\mathcal{O}_y = \mathbb{C}[[y]]$. The variables x and y will be dropped from the notation whenever ambiguity is not likely to arise. We refer to the identity and zero matrices of prescribed dimensions by I_ν and $O_{\nu, \mu}$. System (1) can be rewritten as

$$\begin{cases} x \frac{\partial Y}{\partial x} = A(x, y)Y = x^{-p}(A_0(y) + A_1(y)x + A_2(y)x^2 + \dots)Y \\ y \frac{\partial Y}{\partial y} = B(x, y)Y = y^{-q}(B_0(x) + B_1(x)y + B_2(x)y^2 + \dots)Y \end{cases} \quad (4)$$

where $(A_0(y), B_0(x))$ is called the leading coefficient pair. The Poincaré rank of system (4) is thus (p, q) and we refer to $(A_{00}, B_{00}) := (A(0, 0), B(0, 0))$ as the leading constant pair. An easy calculation shows that the integrability condition is given by

$$x \frac{\partial B}{\partial x} + BA = y \frac{\partial A}{\partial y} + AB. \quad (5)$$

It follows from Theorem 1 that a fundamental matrix of formal solutions has the following form and properties

$$\Phi(x^{1/s_1}, y^{1/s_2}) x^{\Lambda_1} y^{\Lambda_2} \exp(Q_1(x^{-1/s_1})) \exp(Q_2(y^{-1/s_2})) \quad (6)$$

- $s = (s_1, s_2)$ is two-tuple of positive integers;
- Φ is an invertible meromorphic series in $(x^{1/s_1}, y^{1/s_2})$ over \mathbb{C} ;
- Q_1, Q_2 are diagonal matrices containing polynomials in x^{-1/s_1} and y^{-1/s_2} over \mathbb{C} without constant terms; they are obtained by formally integrating the x, y -exponential parts respectively;
- Λ_1 and Λ_2 are constant matrices commuting with Q_1 and Q_2 .

System (4) (resp. system (1)) is said to be *regular singular* whenever Q_1 and Q_2 are null. In this case $s = (1, 1)$, and the formal series ϕ converges whenever the series of A and B do. Otherwise, system (4) is *irregular singular* and the entries of Q_1, Q_2 determine the main asymptotic behavior of the actual solutions as $x_i \rightarrow 0$ in appropriately small sectorial regions (Proposition 5.2, page 232, and Section 4 of [Gérard et al., 1979]). It is shown in [Deligne, 1970; van den Essen, 1979], for the multivariate case in geometric and algebraic settings respectively, that the regularity of system (4) is equivalent to the regularity of the individual subsystems each considered as a system of ordinary differential equations. As a consequence, system (4) is *regular singular* if and only if its true Poincaré rank is $(0, 0)$. And to test its regularity, algorithms given for $m = 1$ (e.g. [Barkatou et al., 2009; Levelt, 1991]) can be applied separately to the individual subsystems.

We end this section by a further characterization of a change of basis T in the bivariate case, which takes system (4) to an *equivalent* system in a weak-triangular form, rather than that of Theorem 1. However, we'll see in Theorem 3 that this form suffices to give an insight into the computation of x, y -exponential parts. The following Proposition is given as [Proposition 3, page 654, [Charrière et al., 1981]] and the proof is to be omitted here due to the lack of space.

Proposition 2 *Consider the completely integrable system (4). Under additional ramification, $x = t^{s_1}$ where s_1 is a positive integer, there exists $T \in GL_n(\mathbb{C}((t, y)))$, product of transformations of type $\text{diag}(t^{k_1}, \dots, t^{k_n})$ (where*

k_1, \dots, k_n are nonnegative integers) and transformations in $GL_n(\mathbb{C}[[t, y]])$, such that the change of basis $T = YZ$ gives the following equivalent system:

$$\begin{cases} \frac{\partial Z}{\partial t} = \tilde{A}(t, y)Z \\ \frac{\partial Z}{\partial y} = \tilde{B}(y)Z \end{cases} \quad (7)$$

where $\tilde{A}(t, y) = \text{Diag}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_h)$ and for all $1 \leq k \leq h$

- $\tilde{A}_k = a_k(t^{-1})I_{n_k} + t^{-1}N_k(y)$;
- $a_k(t) = \sum_{j=1}^{s_1 p+1} \xi_{jk} t^{-j}$ is a polynomial in t^{-1} with coefficients in \mathbb{C} ;
- N_k is a n_k -square matrix with elements in \mathcal{O}_y ;
- $y^{q+1}\tilde{B}(y) \in \mathcal{O}_y$ and $\tilde{B}(y) = \text{Diag}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_h)$;
- $y^{q+1}\tilde{B}_k|_{y=0} = c_k I_{n_k} + F_k$ where $c_k \in \mathbb{C}$ and F_k is a nilpotent constant matrix.

Remark 1 Let $\bar{a}_k(t) = \sum_{j=2}^{s_1 p+1} \xi_{jk} t^{-j}$ then they are the x -exponential parts of system (7) (resp. system (4)).

3 Formal Reduction of ODS

For $m = 1$, system (1) reduces to the linear singular differential system of ordinary equations

$$x \frac{dY}{dx} = A(x)Y = x^{-p}(A_0 + A_1 x + A_2 x^2 + \dots)Y \quad (8)$$

where $A_0 := A(0)$ is the leading coefficient matrix and p is a nonnegative integer denoting the *Poincaré rank*. For a fundamental matrix of solutions, in analogy to (6), we write

$$\Phi(x^{1/s}) x^\Lambda \exp(Q(x^{-1/s})). \quad (9)$$

If $T \in GL_n(\mathbb{C}((x)))$ then the change of basis $Y = TZ$ results in the equivalent system

$$x \frac{dZ}{dx} = \tilde{A}(x)Z = x^{-\tilde{p}}(\tilde{A}_0 + \tilde{A}_1 x + \tilde{A}_2 x^2 + \dots)Z. \quad (10)$$

The second author of this article developed in [Barkatou, 1997] a recursive algorithmic process that constitutes of finding, at every step, a change of basis which results in a system (10) *equivalent* to system (8) which is either of lower *Poincaré rank* or can be decoupled into systems of lower dimensions. At each step, the starting point is the nature of eigenvalues of the leading coefficient matrix A_0 according to which either a block-diagonalized equivalent system, a Moser-irreducible one, or the Katz invariant are computed. The following operations summarize the recursive process:

- **Block-diagonalization:** Whenever A_0 has at least two distinct eigenvalues, system (8) can be decoupled into systems of lower dimensions via the classical Splitting Lemma. For its statement and a constructive proof one may consult [Section 12, pages 52-54, [Wasow, 2002]].
- **Eigenvalue shifting:** Whenever A_0 has a single nonzero eigenvalue $\gamma \in \mathbb{C}$, the so-called Eigenvalue shifting

$$Y = \exp\left(\int \gamma x^{-p-1} dx\right) Z, \quad (11)$$

results in a system with a nilpotent leading coefficient matrix (in fact, $\tilde{A}(x) = A - x^{-p}\gamma I_n$).

- **Moser-based rank reduction:** This is the rank reduction based on the reduction criterion defined by Moser in [Moser, 1960]. It results in an equivalent system whose *Poincaré rank* is the *true Poincaré rank* and whose leading coefficient matrix has the minimal rank among any possible choice of a change of basis. For an efficient Moser-based rank reduction algorithm, one may consult [Barkatou et al., 2009]. If the leading matrix coefficient of the *equivalent* Moser-irreducible system is still nilpotent, one proceeds to compute the Katz invariant, a process for which Moser-irreducibility is a prerequisite.
- **Katz invariant:**

Definition 1 *Given system (8) whose x -exponential parts are denoted by $\{\bar{w}_k\}_{1 \leq k \leq n}$. For $1 \leq k \leq n$, let ζ_k be the minimum exponent in x within the terms of \bar{w}_k . The Katz invariant of (8) (resp. $A(x)$) is then the rational number*

$$\kappa = -1 - \min_{1 \leq k \leq n} \zeta_k.$$

Katz invariant can be obtained from the characteristic polynomial of $A(x)$, i.e. $\det(\lambda I - A(x))$, given that $A(x)$ is Moser-irreducible [e.g. Theorem 1 in [Barkatou, 1997]]. Consider a Moser-irreducible system (8) whose leading matrix coefficient is nilpotent and $\kappa = \frac{l}{m}$ with l, m relatively prime positive integers. Then, a ramification, that is to say a re-adjustment of the independent variable $t = x^{1/m}$, followed by Moser-based rank reduction results in an *equivalent* Moser-irreducible system whose *Poincaré rank* is equal to l and its leading matrix coefficient has at least m distinct eigenvalues. Hence, block-diagonalization may be applied again.

This recursive process results either in a group of decoupled systems with dimension $n = 1$ (scalar case) or *Poincaré rank* $p = 0$ (system with singularity of first kind). For the latter case one may consult Chapter 1 in [Wasow,

2002] or [Barkatou et al., 1999] for a more general context. The changes of basis applied at every stage are used to construct a fundamental matrix of solutions (9). Algorithms of the four described operations are implemented in Maple (see ISOLDE [Barkatou et al., 2013]).

4 Formal Reduction in the bivariate case

Consider again the completely integrable system (4)

$$\begin{cases} x \frac{\partial Y}{\partial x} = A(x, y)Y = x^{-p}(A_0(y) + A_1(y)x + \dots)Y \\ y \frac{\partial Y}{\partial y} = B(x, y)Y = y^{-q}(B_0(x) + B_1(x)y + \dots)Y. \end{cases}$$

A major difficulty within the symbolic manipulation of system (4) (resp. system (1)) arises from (3) as it is evident that any transformation applied to any of the subsystems alters the others. Hence, the generalization of the univariate-case techniques is not straightforward. In particular, the equivalent system does not necessarily inherit the normal crossings even for very simple examples, as exhibited by Example 1 in [Section 4, [Barkatou et al., 2006]] which we recall here.

Example 1 *Consider the following completely integrable pfaffian system with normal crossings of Poincaré rank $(3, 1)$ and true Poincaré rank $(0, 0)$.*

$$\begin{cases} x \frac{\partial Y}{\partial x} = A(x, y)Y = x^{-3} \begin{bmatrix} x^3 + y & y^2 \\ -1 & -y + x^3 \end{bmatrix} Y \\ y \frac{\partial Y}{\partial y} = B(x, y)Y = y^{-1} \begin{bmatrix} y & y^2 \\ -2 & -3 \end{bmatrix} Y. \end{cases} \quad (12)$$

The change of basis $Y = \begin{bmatrix} x^3 & -y^2 \\ 0 & y \end{bmatrix} Z$ computed by the univariate-case Moser-based rank reduction algorithm, upon regarding the first subsystem as an ODS in x , results in the following equivalent system

$$\begin{cases} x \frac{\partial Z}{\partial x} = \tilde{A}(x, y)Z = \begin{bmatrix} -2 & 0 \\ \frac{-1}{y} & 1 \end{bmatrix} Z \\ y \frac{\partial Z}{\partial y} = \tilde{B}(x, y)Z = y^{-2} \begin{bmatrix} -y^2 & 0 \\ -2x^3 & -2y^2 \end{bmatrix} Z. \end{cases}$$

We can see that such a transformation achieves the goal of diminishing the rank of the first subsystem, considered as an ODS, to its minimum ($\tilde{p} = 0$). However, it alters the normal crossings as it introduces the factor y in the denominator of an entry in \tilde{A} . Moreover, it elevates the rank of the second subsystem.

The urge to preserve the normal crossings, whose importance is highlighted in the Introduction, motivates the following definition

Definition 2 *Let $T \in GL_n(K)$. We say that the change of basis $Y = TZ$ (resp. T) is compatible with system (4) if the normal crossings of the system is preserved and the Poincaré rank of the individual subsystems is not elevated.*

Remark 2 *Clearly, if T is a constant matrix or it lies in $GL_n(\mathcal{O})$ then it is compatible with system (4).*

As in the univariate case, the main difficulties in formal reduction arise whenever the leading constant pair consists of nilpotent matrices. Since otherwise a block diagonalization can be attained. A generalization of the univariate-case Splitting Lemma is given with constructive proof in [Section 5.2, page 233, [Gérard et al., 1979]]. We repeat hereby the theorem without proof.

Theorem 2 *Given system (4) with leading constant pair*

$$A_{00} = \text{diag}(A_{00}^1, A_{00}^2) \text{ and } B_{00} = \text{diag}(B_{00}^1, B_{00}^2).$$

If the matrices in one of the couples (A_{00}^1, A_{00}^2) or (B_{00}^1, B_{00}^2) have no eigenvalues in common, then there exists a unique transformation $T \in GL_n(\mathcal{O})$ partitioned conformally

$$T(x, y) = \begin{bmatrix} I & T^{12} \\ T^{21} & I \end{bmatrix}$$

such that the change of basis $Y = TZ$ results in the following equivalent system partitioned conformally with A_{00}, B_{00} ,

$$\begin{cases} x \frac{\partial Z}{\partial x} = \tilde{A}(x, y)Z = \text{diag}(\tilde{A}^1, \tilde{A}^2)Z \\ y \frac{\partial Z}{\partial y} = \tilde{B}(x, y)Z = \text{diag}(\tilde{B}^1, \tilde{B}^2)Z \end{cases}$$

where $(\tilde{A}_{00}^1, \tilde{A}_{00}^2) = (A_{00}^1, A_{00}^2)$ and $(\tilde{B}_{00}^1, \tilde{B}_{00}^2) = (B_{00}^1, B_{00}^2)$.

By an eigenvalue shifting (11) we can then arrive at an equivalent system whose leading constant pair consists of nilpotent matrices. It can be easily verified that such a transformation is *compatible* with system (4). By Proposition 1, the matrices of the leading coefficient pair are nilpotent as well. The case of $n = 1$ is straightforward and the case of $(p, q) = (0, 0)$ is already resolved in [Chapter 3, [LeRoux, 2006]]. Henceforth, following Section 3, the problem of formal reduction is reduced now to discussing two operations: computing Katz invariant and a *compatible* Moser-based rank reduction. The former is the subject of the next section.

5 Computing Exponential Parts and Katz Invariant

In analogy to Definition 1, the Katz invariant of system (4) is defined as follows.

Definition 3 *Given system (4) whose x, y -exponential parts are denoted by $\{\bar{a}_k\}_{1 \leq k \leq n}$ and $\{\bar{b}_k\}_{1 \leq k \leq n}$ respectively. For $1 \leq k \leq n$, let ζ_k (resp. η_k) be the minimum exponent in x (resp. y) within the terms of \bar{a}_k (resp. \bar{b}_k). The Katz invariant of (4) is then the two-tuple of rational numbers (κ_1, κ_2) where*

$$\begin{cases} \kappa_1 = -1 - \min_{1 \leq k \leq n} \zeta_k \\ \kappa_2 = -1 - \min_{1 \leq k \leq n} \eta_k. \end{cases}$$

In this section, we show that the x, y -exponential parts of system (4) are those of the two associated ODS defined below. Hence, the computation of the x, y -exponential parts (consequently of Q_1 , Q_2 , and Katz invariant) is reduced to computations over a univariate field.

Definition 4 *Given system (4). Let $\mathbf{A}(x) := A(x, 0)$ and $\mathbf{B}(y) := B(0, y)$. We call the following the associated ODS with the first and second subsystem respectively of (4):*

$$x \frac{dY}{dx} = \mathbf{A}(x) Y \tag{13}$$

$$y \frac{dY}{dy} = \mathbf{B}(y) Y \tag{14}$$

Theorem 3 *Given the completely integrable system (4)*

$$\begin{cases} x \frac{\partial Y}{\partial x} = A(x, y) Y = x^{-p} (A_0(y) + A_1(y)x + \dots) Y \\ y \frac{\partial Y}{\partial y} = B(x, y) Y = y^{-q} (B_0(x) + B_1(x)y + \dots) Y. \end{cases}$$

The x, y -exponential parts of this system are those of its associated ODS (13) and (14) respectively.

Proof. We prove the theorem for $A(x, y)$. The same follows for $B(x, y)$ by interchanging the order of the subsystems in system (4). Let s_1, t and T be as in Proposition 2. Upon the change of independent variable $x = t^{s_1}$, the first subsystem of (4) is given by

$$\frac{\partial Y}{\partial t} = s_1 t^{-1} A(t^{s_1}, y) Y. \tag{15}$$

By change of basis $Y = TZ$ we arrive at the equivalent subsystem

$$\frac{\partial Y}{\partial t} = \tilde{A}(t, y) Y \tag{16}$$

with the notations and properties as in Proposition 2.

It follows from (3) that

$$\frac{\partial T}{\partial t} = s_1 t^{-1} A T - T \tilde{A}. \quad (17)$$

On the other hand, we have the following expansions as formal power series in y whose matrix coefficients lie in $\mathbb{C}((t))^{n \times n}$:

$$A(t, y) = \sum_{j=0}^{\infty} A_j(t) y^j, \quad \tilde{A}(t, y) = \sum_{j=0}^{\infty} \tilde{A}_j(t) y^j,$$

$$\text{and } T(t, y) = \sum_{j=0}^{\infty} T_j(t) y^j \quad (18)$$

with leading terms $\mathbf{A}(t^{s_1})$, $\tilde{\mathbf{A}}$, and \mathbf{T} respectively. We Plug (18) in (17) and compare the like-power terms. In particular, we are interested in the relation between the leading terms which is clearly given by

$$\frac{\partial \mathbf{T}}{\partial t} = s_1 t^{-1} \mathbf{A} \mathbf{T} - \mathbf{T} \tilde{\mathbf{A}}. \quad (19)$$

Due to the form of $T(t, y)$ characterized in Theorem 2, it is evident that $\mathbf{T} \in GL_n(\mathbb{C}((t)))$. Hence, the systems given by $\frac{\partial}{\partial t} - s_1 t^{-1} \mathbf{A}(t^{s_1})$ (resp. $x \frac{\partial}{\partial x} - \mathbf{A}(x)$) and $\frac{\partial}{\partial t} - \tilde{\mathbf{A}}$ are *equivalent*. It follows that they have the same *x-exponential parts* $\bar{a}_k(t)$ given in Remark 1. ■

Two corollaries follow directly from Theorem 3.

Corollary 1 *Let $\kappa = (\kappa_1, \kappa_2)$ be as in Definition 3 denoting the Katz invariant of system (4). Then κ_1 (resp. κ_2) is the Katz invariant of the associate ODS (13) (resp. (14)).*

Corollary 2 *Let $\gamma = (\gamma_1, \gamma_2)$ denote the true Poincaré rank of system (4). Then γ_1 (resp. γ_2) is the true Poincaré rank of the associate ODS (13) (resp. (14)).*

Proof. Consider system (13). By (Remark 3, [Barkatou, 1997]) we have $\gamma_1 - 1 \leq \kappa_1 \leq \gamma_1$. This fact establishes the proof since γ_1 is an integer. The same holds for system (14). ■

Hence, the *Katz invariant*, the *true Poincaré rank*, and most importantly Q_1, Q_2 in (6), can be computed efficiently over univariate fields using the existing package ISOLDE [Barkatou et al., 2013]. Moreover, since now we know κ at any stage of formal reduction, it is left to give a *compatible* Moser-based rank reduction of system (4). We remark however, that although the rank reduction of [Barkatou et al., 2006] reduces the *Poincaré rank* to its minimal integer value, Moser-based rank reduction results in an *equivalent* system for which the *Poincaré rank* and the rank of the leading matrix coefficient are both minimal. In the univariate case, this is a necessary component of formal reduction, in particular when computing the *Katz invariant*.

6 Moser-based Rank Reduction

We consider again system (4) and label its subsystems as follows:

$$x \frac{\partial Y}{\partial x} = A(x, y)Y = x^{-p}(A_0(y) + A_1(y)x + \dots)Y \quad (20)$$

$$y \frac{\partial Y}{\partial y} = B(x, y)Y = y^{-q}(B_0(x) + B_1(x)y + \dots)Y. \quad (21)$$

Let $T \in GL_n(K)$. To keep track of applied transformations, we use the following notation for the *equivalent* system resulting upon the change of basis $Y = TZ$.

$$x \frac{\partial Z}{\partial x} = \tilde{A}Z, \quad T[A] := \tilde{A} = T^{-1}(AT - x \frac{\partial T}{\partial x}) \quad (22)$$

$$y \frac{\partial Z}{\partial y} = \tilde{B}Z, \quad T[B] := \tilde{B} = T^{-1}(BT - y \frac{\partial T}{\partial y}). \quad (23)$$

In particular, $T^{-1}AT$ will be referred to as the *similarity term* of $T[A]$.

We study Moser-based rank reduction of subsystem (20). The same results follow for subsystem (21) by interchanging (20) its order in system (4). We adapt the algorithm given for singularly-perturbed linear differential systems in [Barkatou et al., 2014] since it is well-suited to bivariate fields. It suffices to verify that the transformations in the proposed algorithm of system (20) are compatible with the second. We will see that this can be guaranteed by giving an additional structure to $A_0(y)$, in particular form (24), rather than the form proposed in (Lemma 1, [Barkatou et al., 2014]). We remark that the algorithm of [Barkatou et al., 2014] is a generalization of Moser-based rank reduction developed by the second author of this article in [Barkatou, 1995] over a univariate field. We recall that, in the sequel, we drop x and y from the notation whenever ambiguity is unlikely to arise. Set $r = \text{rank}(A_0(y))$.

Following [Barkatou, 1995; Moser, 1960] we define the Moser rank and Invariant of system (20) as the respective rational numbers:

$$m(A) = \max(0, p + \frac{r}{n})$$

$$\mu(A) = \min \{m(T[A]) | T \in GL_n(K)\}.$$

Definition 5 *The system (20) (the matrix A respectively) is called Moser-reducible if $m(A) > \mu(A)$, otherwise it is said to be Moser-irreducible.*

It is easy to see from this definition that system (20) is regular if and only if $\mu(A) \leq 1$, i.e. the *true Poincaré rank* is zero.

Definition 6 *System (4) is said to be Moser-irreducible whenever each of its subsystems (20) and (21) is.*

The following theorem is the analog of (Theorem 1, [Barkatou et al., 2014]).

Theorem 4 *Given System (20) such that $A_0(y)$ is of rank r and $m(A) > 1$. A necessary and sufficient condition for A to be Moser-reducible, i.e. for the existence of a $T(x, y) \in GL_n(K)$ such that $r(\tilde{A}_0) < r$, is that the polynomial*

$$\theta(\lambda) := x^r \det(\lambda I + \frac{A_0}{x} + A_1)|_{x=0}$$

vanishes identically in λ . Moreover, $T(x, y)$ can always be chosen to be compatible with system (21). More precisely, it is a product of transformations in $GL_n(\mathcal{O}_y)$ and polynomial transformations of the form $\text{diag}(x^{\alpha_1}, \dots, x^{\alpha_n})$ where $\alpha_1, \dots, \alpha_n$ are nonnegative integers.

Remark 3 *By Corollary 2, the true Poincaré rank of system (4) can be deduced from its associated ODS. However, the reduction criterion in Theorem 4 guarantees that the rank of the leading coefficient matrix of the equivalent Moser-irreducible system is minimum as well, not only its Poincaré rank. Moreover, this criterion furnishes the construction of the change of basis $T(x, y)$ as will be demonstrated in the sequel.*

Theorem 4 is to be proved after giving its necessary building blocks. We start by the following lemma.

Lemma 1 *There exists a unimodular transformation $U(y) \in GL_n(\mathcal{O}_y)$ such that for the resulting equivalent system (22), we have*

$$\tilde{A}_0(y) = \begin{bmatrix} \tilde{A}_0^{11} & O & O \\ \tilde{A}_0^{21} & O_{r-d} & O \\ \tilde{A}_0^{31} & \tilde{A}_0^{32} & O_{n-r} \end{bmatrix} \quad (24)$$

where

$$\begin{bmatrix} \tilde{A}_0^{11} \\ \tilde{A}_0^{21} \\ \tilde{A}_0^{31} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{A}_0^{11} & O \\ \tilde{A}_0^{21} & O_{r-d} \\ \tilde{A}_0^{31} & \tilde{A}_0^{32} \end{bmatrix}$$

are $r \times d$ and $n \times r$ matrices of full column ranks d and r respectively.

Proof. It suffices to apply the unimodular transformation of (Lemma 1, [Barkatou et al., 2014]) to $A_0(y)$ and then to the first left block of the resulting similar matrix. Denoting respectively by $U_1(y), U_2(y)$ these transformations, we set

$U(y) = \text{diag}(U_2(y), I_{n-r}).U_1(y)$. Hence, the leading coefficient matrix of $U[A]$ has the form (24). Clearly, $U(y)$ is *compatible* with system (21) as it is unimodular. ■

Hence, without loss of generality, we assume that $A_0(y)$ is in form (24) and partition $A_1(y)$ conformally with A_0 . Let

$$G_\lambda(A) = \begin{bmatrix} A_0^{11} & O & A_1^{13} \\ A_0^{21} & O & A_1^{23} \\ A_0^{31} & A_0^{32} & A_1^{33} + \lambda I_{n-r} \end{bmatrix}. \quad (25)$$

Then we have the following Lemma given and proved as (Lemma 2, [Barkatou et al., 2014]).

Lemma 2 *Det($G_\lambda(A)$) $\equiv 0$ vanishes identically in λ if and only if $\theta(\lambda)$ does.*

Proposition 3 *Suppose that $m(A) > 1$ and $\det(G_\lambda(A)) \equiv 0$ where $G_\lambda(A)$ is given by (25). Then there exists a unimodular matrix $Q(y)$ in $GL_n(\mathcal{O}_y)$ with $\det Q(y) = \pm 1$, compatible with system (21), such that the matrix $G_\lambda(Q[A])$ has the form*

$$G_\lambda(Q[A]) = \begin{bmatrix} A_0^{11} & O & U_1^{11} & U_2^{11} \\ A_0^{21} & O & U_1^{21} & U_2^{21} \\ V_1^{11} & V_1^{12} & W_1 + \lambda I_{n-r-\rho} & W_2 \\ M_1^{11} & M_1^{12} & M_2 & W_3 + \lambda I_\rho \end{bmatrix}, \quad (26)$$

where $0 \leq \rho \leq n - r$, W_1 , W_3 are square matrices of orders $(n - r - \rho)$ and ρ respectively, M_1^{12} is a null matrix, and

$$\text{rank} \begin{bmatrix} A_0^{11} & U_1^{11} \\ A_0^{21} & U_1^{21} \\ M_1^{11} & M_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_0^{11} & U_1^{11} \\ A_0^{21} & U_1^{21} \end{bmatrix}, \quad (27)$$

$$\text{rank} \begin{bmatrix} A_0^{11} & U_1^{11} \\ A_0^{21} & U_1^{21} \end{bmatrix} < r. \quad (28)$$

Proof. Since $Q(y)$ is independent form x , it follows from (22) that the discussion can be restricted to the similarity term of the transformations. Hence, the transformation $Q(y)$ can be constructed as in the proof of (Proposition 2, [Barkatou et al., 2014]). As it is unimodular, it is necessarily *compatible* with system (21). It remains to remark however, that each row of $[M_1^{11} \ M_1^{12} \ M_2]$ is a linear combination of the rows of

$$\begin{bmatrix} A_0^{11} & O & U_1^{11} \\ A_0^{21} & O & U_1^{21} \end{bmatrix}.$$

Hence, by construction, M_1^{12} is a null matrix. ■

Proposition 4 *If $m(A) > 1$ and $G_\lambda(A) \equiv 0$ is as in (26) with conditions (27) and (28) satisfied, then system (20) (resp. A) is Moser-reducible and reduction can be carried out with the so-called shearing $Y = SZ$ where $S = \text{diag}(xI_r, I_{n-r-\rho}, xI_\rho)$ if $\rho \neq 0$ and $S = \text{diag}(xI_r, I_{n-r})$ otherwise. Furthermore, this shearing is compatible with system (21).*

Proof. We partition $A(x, y)$ conformally with (26)

$$A = \begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ A^{21} & A^{22} & A^{23} & A^{24} \\ A^{31} & A^{32} & A^{33} & A^{34} \\ A^{41} & A^{42} & A^{43} & A^{44} \end{bmatrix}$$

where $A^{11}, A^{22}, A^{33}, A^{44}$ are of dimensions $d, r-d, n-r-\rho, \rho$ respectively. It is easy to verify then, by (22) and (23), that

$$\begin{aligned} \tilde{A}(x, y) &= S^{-1}AS - xS^{-1}\frac{\partial}{\partial x}S = \begin{bmatrix} A^{11} & A^{12} & x^{-1}A^{13} & A^{14} \\ A^{21} & A^{22} & x^{-1}A^{23} & A^{24} \\ xA^{31} & xA^{32} & A^{33} & xA^{34} \\ A^{41} & A^{42} & x^{-1}A^{43} & A^{44} \end{bmatrix} \\ &\quad - \text{diag}(I_r, O_{n-r-\rho}, I_\rho) \\ \tilde{B}(x, y) &= S^{-1}BS = \begin{bmatrix} B^{11} & B^{12} & x^{-1}B^{13} & B^{14} \\ B^{21} & B^{22} & x^{-1}B^{23} & B^{24} \\ xB^{31} & xB^{32} & B^{33} & xB^{34} \\ B^{41} & B^{42} & x^{-1}B^{43} & B^{44} \end{bmatrix}. \end{aligned}$$

Hence, the new leading coefficient matrix is

$$\tilde{A}_0 = \begin{bmatrix} A_0^{11} & O & U_1^{11} & O \\ A_0^{21} & O & U_1^{21} & O \\ O & O & O & O \\ M_1^{11} & O & M_2 & O \end{bmatrix}$$

where $\text{rank}(\tilde{A}_0) < r$ since (27) and (28) are satisfied.

It remains to prove the compatibility of S with the subsystem (21), in particular, that the normal crossings is preserved. It suffices to prove that the submatrices of $B(x, y)$ which are multiplied by x^{-1} , i.e. B^{13}, B^{23}, B^{43} , have no term independent from x so that no poles in x are introduced. This can be restated as requiring $B^{13}(0, y), B^{23}(0, y)$, and $B^{43}(0, y)$ to be null. This requirement is always satisfied due to the integrability condition (5). In fact, we can obtain from the former that

$$B(0, y)A_0(y) - A_0(y)B(0, y) = y\frac{\partial A_0(y)}{\partial y}. \quad (29)$$

On the other hand, since $G_\lambda(A)$ is as in (26) then $A_0(y)$ have the following form (30) and $B(0, y)$ can be partitioned conformally

$$A_0(y) = \begin{bmatrix} A_0^{11} & O & O & O \\ A_0^{21} & O & O & O \\ V_1^{11} & V_1^{12} & O & O \\ M_1^{11} & O & O & O \end{bmatrix}. \quad (30)$$

$$B(0, y) = \begin{bmatrix} B_{0y}^{11} & B_{0y}^{12} & B_{0y}^{13} & B_{0y}^{14} \\ B_{0y}^{21} & B_{0y}^{22} & B_{0y}^{23} & B_{0y}^{24} \\ B_{0y}^{31} & B_{0y}^{32} & B_{0y}^{33} & B_{0y}^{34} \\ B_{0y}^{41} & B_{0y}^{42} & B_{0y}^{43} & B_{0y}^{44} \end{bmatrix}. \quad (31)$$

Inserting (30) and (31) in (29), one can obtain the desired results by equating the entries of (29). In particular, upon investigating the entries of the L.H.S. in (Column 3), (Rows 1 and 2, Column 2), and (Row 4, Column 2), we observe the following respectively:

- $\begin{bmatrix} A_0^{11} & O \\ A_0^{21} & O \\ V_1^{11} & V_1^{12} \\ M_1^{11} & O \end{bmatrix} \begin{bmatrix} B_{0y}^{13} \\ B_{0y}^{23} \end{bmatrix} = O_{n, n-\rho-r}$. The first matrix is of full rank r by construction thus $\begin{bmatrix} B_{0y}^{13} \\ B_{0y}^{23} \end{bmatrix}$ is null.
- $\begin{bmatrix} A_0^{11} \\ A_0^{21} \end{bmatrix} B_{0y}^{12} = O_{r, r-d}$. The first matrix is of full rank d by construction thus B_{0y}^{12} is null.
- Finally, $B_{0y}^{43} V_1^{12} - M_1^{11} B_{0y}^{12} = O_{\rho, (r-d)}$. But B_{0y}^{12} is null and V_1^{12} is of full column rank $r - d$ by construction and so B_{0y}^{43} is null as well.

This completes the proof. ■

We give hereby the proof of Theorem 4.

Proof. (Theorem 4) For the necessary condition, we proceed as in the proof of Theorem 1 in [Barkatou et al., 2014]. As for the sufficiency, we set $r = \text{rank}(A_0(y))$. Without loss of generality, we can assume that $A_0(y)$ has the form (24). Let $G_\lambda(A)$ be given as in (25). Then, by Lemma 2, $\det(G_\lambda(A))$ vanishes identically in λ if and only if $\theta(\lambda)$ does. Then the matrix $S[Q[A]]$ where S, Q are as in Propositions 3 and 4 respectively, has the desired property. ■

Corollary 3 *Given system (4) with Poincaré rank (p, q) . A necessary and sufficient for it to be Moser-reducible is that one of the polynomials*

$$\begin{cases} \theta_A(\lambda) := x^{\text{rank}(A_0)} \det(\lambda I + \frac{A_0}{x} + A_1)|_{x=0} \\ \theta_B(\lambda) := y^{\text{rank}(B_0)} \det(\lambda I + \frac{B_0}{y} + B_1)|_{y=0} \end{cases}$$

vanishes identically in λ . An equivalent Moser-irreducible system can be attained via a change of basis compatible with both subsystems (20) and (21).

We illustrate the process by this simple example.

Example 2

$$\begin{cases} x \frac{\partial Y}{\partial x} = x^{-3} \begin{bmatrix} x^3 + x^2 + y & y^2 \\ -1 & x^3 + x^2 - y \end{bmatrix} Y \\ y \frac{\partial Y}{\partial y} = y^{-2} \begin{bmatrix} y^2 - 2y - 6 & y^3 \\ -2y & -3y^2 - 2y - 6 \end{bmatrix} Y \end{cases} \quad (32)$$

The associated ODS are then respectively:

$$\begin{cases} x \frac{\partial Y}{\partial x} = x^{-3} \begin{bmatrix} x^3 + x^2 & 0 \\ -1 & x^3 + x^2 \end{bmatrix} Y, \\ y \frac{\partial Y}{\partial y} = y^{-2} \begin{bmatrix} y^2 - 2y - 6 & y^3 \\ -2y & -3y^2 - 2y - 6 \end{bmatrix} Y. \end{cases}$$

Via ISOLDE, we compute $Q_1 = \frac{-1}{x}I_2$ and $Q_2 = \frac{3}{y^2} + \frac{2}{y}I_2$. Thus we have $s_1 = s_2 = 1$ and (6) is given by

$$\Phi(x, y) x^{\Lambda_1} y^{\Lambda_2} e^{\frac{-1}{x}} e^{\frac{3}{y^2} + \frac{2}{y}}.$$

Upon the eigenvalue shifting $Y = e^{\frac{-1}{x}} e^{\frac{3}{y^2} + \frac{2}{y}} Z$ we get from (32)

$$\begin{cases} x \frac{\partial Z}{\partial x} = x^{-3} \begin{bmatrix} x^3 + y & y^2 \\ -1 & x^3 - y \end{bmatrix} Z \\ y \frac{\partial Z}{\partial y} = y^{-1} \begin{bmatrix} y & y^2 \\ -2 & -3y \end{bmatrix} Z \end{cases}$$

We arrive at the system of Example (1). By Algorithm 1, we compute $T_1 = \begin{bmatrix} yx^3 & -y \\ 0 & 1 \end{bmatrix}$. Hence, by $Z = T_1 U$, we have

$$\begin{cases} x \frac{\partial U}{\partial x} = \begin{bmatrix} -2 & 0 \\ -y & 1 \end{bmatrix} U \\ y \frac{\partial U}{\partial y} = \begin{bmatrix} -2 & 0 \\ -2x^3 & -1 \end{bmatrix} U \end{cases}$$

By a simple calculation, we find $T_2 = \begin{bmatrix} 1 & 0 \\ \frac{y}{3} + 2x^3 & -1 \end{bmatrix}$.

A fundamental matrix of solutions is then given by

$$T_1 T_2 x^{\Lambda_1} y^{\Lambda_2} e^{\frac{-1}{x}} e^{\frac{3}{y^2} + \frac{2}{y}}$$

where $\Lambda_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Lambda_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$.

Algorithm 1 Moser-based Rank Reduction of System (4)

Input: $A(x, y), B(x, y)$ of (4)

Output: $T(x, y)$ a change of basis and a Moser-irreducible equivalent system $\{T[A], T[B]\}$. In particular, the Poincaré rank of this system is its true Poincaré rank.

```
 $T \leftarrow I_n; p \leftarrow \text{Poincaré rank of } A;$ 
 $U(y) \leftarrow \text{Lemma 1};$ 
 $A \leftarrow U^{-1}AU; T \leftarrow TU;$ 
while  $\text{Det}(G_\lambda(A) = 0)$  and  $p > 0$  do
     $Q(y), \rho \leftarrow \text{Proposition 3};$ 
     $S(x) \leftarrow \text{Proposition 4};$ 
     $P \leftarrow QS; T \leftarrow TP;$ 
     $A \leftarrow P^{-1}AP - xS^{-1}\frac{\partial S}{\partial x};$ 
     $p \leftarrow \text{Poincaré rank of } A;$ 
     $U(y) \leftarrow \text{Lemma 1};$ 
     $A \leftarrow U^{-1}AU; T \leftarrow TU;$ 
end while.
 $B \leftarrow T^{-1}BT - yT^{-1}\frac{\partial T}{\partial y};$ 
 $T \leftarrow I_n; q \leftarrow \text{Poincaré rank of } B;$ 
 $U(x) \leftarrow \text{Lemma 1};$ 
 $B \leftarrow U^{-1}BU; T \leftarrow TU;$ 
while  $\text{Det}(G_\lambda(B) = 0)$  and  $q > 0$  do
     $Q(x), \rho \leftarrow \text{Proposition 3};$ 
     $S(y) \leftarrow \text{Proposition 4};$ 
     $P \leftarrow QS; T \leftarrow TP;$ 
     $B \leftarrow P^{-1}BP - yS^{-1}\frac{\partial S}{\partial y};$ 
     $q \leftarrow \text{Poincaré rank of } B;$ 
     $U(x) \leftarrow \text{Lemma 1};$ 
     $B \leftarrow U^{-1}BU; T \leftarrow TU;$ 
end while.
 $A \leftarrow T^{-1}AT - xT^{-1}\frac{\partial T}{\partial x};$ 
return (T, A, B).
```

7 Conclusion and Further Investigations

We gave an explicit method to compute the x, y -exponential parts of a *completely integrable pfaffian system with normal crossings in two variables*. This gives the main information about the asymptotic behavior of its solutions. Moreover, this new approach limited the computations to a finite number of constant matrices instead of matrix-valued functions and constituted an eminent portion of the formal reduction. To complement our work, we gave a Moser-based rank reduction algorithm. Both results, allowed us to generalize the formal reduction of the univariate case as developed in [Barkatou, 1997] to the bivariate case. One field to investigate for this bivariate system would be an algorithm to construct a basis for the space of regular solutions (see, e.g., [Barkatou et al., 1999, 2013], for $m = 1$).

Another research direction over bivariate fields is the generalization of techniques developed here to *completely integrable pfaffian systems* with no restriction to the locus of singularities (i.e. general crossings rather than normal crossings). Such systems are discussed e.g. in [Novikov et al., 2002].

And there remains of course the ultimate task of formal reduction of the multivariate system with no restriction to the number of variables. Theorem 1 was first given and proved in [Charrière, 1980] for bivariate systems. In the theory developed there, one operator Δ_i was considered and the fact that \mathcal{O}_y and \mathcal{O}_x are principal ideal domains was used in many places to prove that certain modules introduced are free modules. This did not allow an immediate generalization to the case of more than two variables. The same obstacle arises in *rank reduction* whether in [Barkatou et al., 2006] or in adapting the Moser-based rank reduction algorithm of [Barkatou et al., 2014; Barkatou, 1995]. This limits our proposed formal reduction in Section 6 to $m = 2$. However a generalization of Proposition 2 is Theorem 2.3 in [Charrière et al., 1981]. This furnishes the generalization of Theorem 3 to a general multivariate system. Furthermore, it motivates the investigation of their formal reduction since the leading coefficient matrix of a change of basis which takes the system to a weak-triangular form, would be characterized.

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